Non-Commutative Formal Groups in Positive Characteristic

Frederick Leitner

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Abstract

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In characterizing non-commutative formal groups $\mathbb{G}$, it is natural to consider their distribution algebras, $\mathbf{Dist}(\mathbb{G})$. Although we are interested mainly in the characteristic $p > 0$ setting, we make an effort to work by analogy to the characteristic zero version of our problem. Let $\mathbb{G}$ be a formal group with Lie algebra $\mathfrak{g}$. We recall that, in characteristic zero, we have a canonical isomorphism between $\mathbf{Dist}(\mathbb{G})$ and the universal enveloping algebra $\mathfrak{U}\mathfrak{g}$. Then our problem reduces to describing the algebraic relationship between the symmetric $\mathbf{S}\mathfrak{g}$, $\mathfrak{U}\mathfrak{g}$, and the tensor algebra $\mathbf{T}\mathfrak{g}$ as in the following diagram:

$$\begin{tikzcd} & \mathbf{T}\mathfrak{g} \arrow[dl] \arrow[dr] & \\ \mathbf{S}\mathfrak{g} \arrow[rr, "\sim", "\mathbf{PBW}"'] & & \mathfrak{U}\mathfrak{g}. \end{tikzcd} \tag{1}$$

Here, the diagonal arrows are algebra maps and the horizontal arrow is a vector space isomorphism given by the Poincaré-Birkhoff-Witt (PBW) theorem.

In the positive characteristic setting, there are a few features that present difficulties which are absent in the zero characteristic setting. In a sense made precise below, the problem is non-linear and, unlike in the zero characteristic setting, the algebras of importance are not quadratic.

Let us recall that in characteristic zero, the PBW theorem gives an identification between $\mathbf{S}\mathfrak{g}$ and the associated graded algebra of $\mathfrak{U}\mathfrak{g}$. We may regard $\mathbf{S}\mathfrak{g}$ as the universal enveloping algebra for the trivial Lie algebra structure on $\mathfrak{g}$, so

that $\mathbf{S}\mathfrak{g}$ may be viewed as the distribution algebra of the unique commutative formal group of dimension $n = \dim(\mathfrak{g})$. Indeed, $\mathbf{S}\mathfrak{g} = \mathbf{Dist}(\mathbb{G}_a^n)$ where $\mathbb{G}_a$ is the additive formal group. In fact, one may identify, regardless of the characteristic, the associated graded algebra of $\mathbf{Dist}(\mathbb{G})$ with $\mathbf{Dist}(\mathbb{G}_a^n)$. However, in positive characterisitc the difficulty arises there are many choices of commutative formal groups of a fixed dimension ([3], [10], [14]). Thus, one might suspect that passing to the associated graded algebra is too course as it loses too much of the underlying commutative geometry of the formal group. It is for these reasons that we introduce geometric formal groups and describe Poisson structure which preserve these commutative sub-structures.

In §1, we introduce the notion of "splay" algebras and prove a PBW type theorem for them. In §2, we recall some relevant details of distribution algebras of formal groups. In §3, we characterize completely geometric non-commutative formal groups in terms of certain Poisson structures on their splays.

We enforce several assumptions and notations throughout this paper. By R we will denote a commutative ring with unit, and if it is a field, we will denote it by $k = R$. All algebras are associate unital R-algebras, unless they are Lie algebras. All tensor products will be taken over R. If V is a free R-module, we denote by $\mathbf{T}V$ the tensor algebra on V (over R), and $\mathbf{S}V$ the symmetric algebra on V. If $\mathcal{X}$ is a set, we denote by $\mathbf{T}\mathcal{X}$ the free algebra over R generated by $\mathcal{X}$ and by $\mathbf{S}\mathcal{X}$ the free commutative algebra generated by $\mathcal{X}$. All ideals will be two-sided ideals. If A and B are algebras, we denote by $A \otimes B \xrightarrow{\tau} B \otimes A$ the **transposition** or **swapping** map $a \otimes b \xmapsto{\tau} b \otimes a$. We assume that none of the algebras A under consideration have quasi-zeros, i.e. elements $a \in A$ such that for all $b, c \in A$ one has $bac = 0$. We also make implicit use of the theory of non-commutative Gröbner basis, for which we follow the notation and conventions of [12] in the case of a field k and for general R those in [11]. By abuse of language, non-commutative often means not necessarily commutative.

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1 Poisson Algebras with Internal Symmetry

In this section, we generalize the algebraic relationship as indicated in (Eqn. **1**) for a Lie algebra $\mathfrak{g}$ over R. Choosing any R-module basis $\mathfrak{g} = R\langle X_i \rangle_{i \in I}$, one can view $\mathbf{S}\mathfrak{g}$ as the tensor product of the one dimensional algebras $\mathbf{S}\{X_i\}$. It is from this vantage point that we wish to generalize. We will be looking at S, a commutative algebra, which has a fixed identification as the tensor product $S = \bigotimes_{i \in I} A_i$ of commutative sub-algebras A_i for i in some indexing set I. We then desire a non-commutative algebra U for which there is a PBW-type isomorphism between S and U, and an algebra T for which both U and S are

quotients by appropriate commutator relations:

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ S \arrow[rr, "\sim", "\textbf{PBW}"'] & & U. \end{tikzcd}$$

Specifically, we desire that S be the quotient of T by ideal of relations generated by $fg - gf$ for $f, g \in T$. We recall that a Poisson structure on an algebra A is a skew-symmetric bi-derivation π which satisfies the Jacobi identity.

Definition 1.1.

1.1.1. Let I be an indexing set and for each i fix a commutative algebra A_i. We define the **splay** of the A_i to be the following diagram $\overline{T}$ in the category of algebras:

$$A_i \xhookrightarrow{\iota_i} \coprod_{i \in I} A_i.$$

We call $T := \coprod_i A_i$, the co-product of the A_i, the **splay algebra** of $\overline{T}$ and we call the A_i the **coordinate algebras** of $\overline{T}$.

1.1.2. Let $\overline{T}$ (resp. $\overline{T}'$) be a splay with coordinate algebras A_i for $i \in I$ (resp. B_j for $j \in J$). A **morphism** of splays $\overline{T} \xrightarrow{\Phi} \overline{T}'$ is an algebra morphism $T \xrightarrow{\Phi} T'$ of the respective splay algebras such that if for every $i \in I$ there is some $j_i \in J$ and some $A_i \xrightarrow{\Phi_{j_i}} B_j$ such that Φ is the map induced from the UMP of the co-product T as follows:

$$\begin{tikzcd} A_i \arrow[r, hook, "\iota_i"] \arrow[d, "\Phi_{j_i}"'] & \coprod A_i \arrow[ddl, dashed, "\exists!"'] \\ B_{j_i} \arrow[d, "\iota_{j_i}"'] & \\ \coprod_i B_j & \end{tikzcd}$$

1.1.3. Let π be a Poisson structure on an algebra A, and suppose that the A_i are sub-algebras. Then we say that π has **internal symmetry** (with respect to the A_i) if $\pi(A_i, A_i) = 0$. If $\overline{T}$ is a splay with coordinate algebras A_i, then a Poisson structure π on $\overline{T}$ is a Poisson structure π on T with internal symmetry with respect to the A_i's.

1.1.4. Let T and T' be splay algebras equipped with Poisson structures π and π' respectively. We say that a morphism of splays $\overline{T} \xrightarrow{\Phi} \overline{T}'$ is a **Poisson morphism** if $\Phi \circ \pi = \pi' \circ (\Phi \otimes \Phi)$. If $T = T'$ as splays, then we say that π and π' are **equivalent** Poisson structures and that Φ gives an **equivalence** between π and π'.

1.1.5. Let A be an algebra and π a Poisson structure on A. We say that π **vanishes on constants** if for all $a \in A$ we have $\pi(1, a) = 0 = \pi(a, 1)$.

1.1.6. Suppose that $A = \bigcup_{n \geq 0} A^{(n)}$ is a filtered algebra and π is a Poisson structure on A. We say that π is **strongly filtered** if:

$$\pi(A^{(m)}, A^{(n)}) \subseteq A^{(m+n-1)}.$$

We will assume for the remainder of this paper that all Poisson structures vanish on constants.

Example 1.2. For an algebra A one has the **canonical Poisson structure** defined by:

$$\pi_{\mathbf{c}}(a, b) := ab - ba$$

for $a, b \in A$. The fact that it is a Poisson structure follows from the associativity of the algebra A. As we have assumed that A has no quasi-zeros, clearly $\pi_{\mathbf{c}}$ vanishes if and only if A is a commutative algebra, so that $\pi_{\mathbf{c}}$ is a measure of the non-commutativity of A. ∎

Remark 1.3. Looking ahead, the Poisson structures π on the splay algebras measure the non-commutativity of the algebras U. In our motivating example of $\mathfrak{g}$ one takes $T = \mathbf{T}\mathfrak{g}$, $S = \mathbf{S}\mathfrak{g}$, and $U = \mathfrak{U}\mathfrak{g}$. One usually considers the Kostant-Kirilov-Souriau $\pi_{\mathbf{KKS}}$ Poisson bracket which is a Poisson bracket on $\mathbf{S}\mathfrak{g}$ rather than $\mathbf{T}\mathfrak{g}$ (c.f. [9]). However, one may may also form a Poisson bracket $\tilde{\pi}$ on $\mathbf{T}\mathfrak{g}$ by extending the Lie bracket by the bi-derivation law. Working modulo the commutator relations $\eta\zeta - \zeta\eta$ for $\eta, \zeta \in \mathfrak{g}$ we have for all $f \in \mathbf{T}\mathfrak{g}$ that:

$$\begin{aligned} \tilde{\pi}(\eta\zeta - \zeta\eta, f) &\equiv \eta\tilde{\pi}(\zeta, f) + \tilde{\pi}(\eta, f)\zeta \\ &\quad -\zeta\tilde{\pi}(\eta, f) - \tilde{\pi}(\zeta, f)\eta \\ &\equiv 0 \end{aligned}$$

so that $\tilde{\pi}$ on $\mathbf{T}\mathfrak{g}$ descends to $\pi_{\mathbf{KKS}}$ on $\mathbf{S}\mathfrak{g}$.

We cannot hope to have such a nice situation in general. For example, suppose that R is a $\mathbb{Z}/m\mathbb{Z}$-algebra for some $m \geq 2$, and suppose that we have a relation of the form $f^m = g$ in S. Then in order for a Poisson structure π on T to descend to S we would need for all $h \in T$ that:

$$\begin{aligned} 0 &\overset{\text{REQ}}{\equiv} \pi(f^m - g, h) \\ &\equiv m! f^{m-1}\pi(f, h) - \pi(g, h) \\ &\equiv -\pi(g, h) \end{aligned}$$

so that we must have either $g = 0$ or $\pi(g, h) = 0$. Although this will not happen in general, we will see below that $S = \mathbf{Dist}(\mathbb{G}_a^n)$, over any $\mathbb{F}_p$-algebra R, falls into the former case, and thus a Poisson structure on the splay of some $\mathbf{Dist}(\mathbb{G}_a)$'s will descend. ∎

There are two main types of PBW proofs. One makes use of the theory of Gröbner bases as in [15] and [4]. The other is from the viewpoint of deformations of Koszul algebras as in [1] and [5]. Below, we generalize the Gröbner basis setting.

Remark 1.4.

1.4.1. Let $\mathcal{X}$ be a set and suppose that there is a map $\mathcal{X} \xrightarrow{\mathbf{deg}} \mathbb{Z}_{\geq 1}$. Suppose that $<$ is a total ordering on the set $\mathcal{X}$. Then if A is either $\mathbf{S}\mathcal{X}$ or $\mathbf{T}\mathcal{X}$, we obtain a monomial ordering in the sense of [12] and [11] on A by using the graded lexicographic ordering where for a monomial $\underline{m} = x_1 \cdots x_n$ with $x_i \in \mathcal{X}$, we define its degree and length by:

$$\begin{aligned} \mathbf{deg}(\underline{m}) &:= \sum_{1 \leq i \leq n} \mathbf{deg}(x_i) \\ \mathbf{len}(\underline{m}) &:= n. \end{aligned}$$

We define:

$$A^{(n)} := R\langle \underline{m} \mid \mathbf{deg}(\underline{m}) \leq n \rangle \quad \subseteq A$$

to be the R-span of monomials of degree at most n, we see that A is a filtered algebra. A is in fact graded algebra where:

$$\begin{aligned} A^n &:= R\langle \underline{m} \mid \mathbf{deg}(\underline{m}) = n \rangle \quad \subseteq A \\ A &= \bigoplus_{n \geq 0} A^n. \end{aligned}$$

If, in particular, we take $\mathcal{X} \xrightarrow{\mathbf{deg}} \mathbb{Z}_{\geq 1}$ to be the constant function 1, then $\mathbf{deg}(\underline{m}) = \mathbf{len}(\underline{m})$ and we recover the usual grading on $\mathbf{S}\mathcal{X}$ and $\mathbf{T}\mathcal{X}$. If $\mathcal{J}$ is an ideal in A and $B := A/\mathcal{J}$, then B inherits a filtration from A:

$$B = \bigcup_{n \geq 0} B^{(n)} \qquad B^{(n)} := q(A^{(n)})$$

where $A \xrightarrow{q} B$ is the quotient map. We will assume below that all algebras constructed in this manner are equipped with this filtration.

1.4.2. Let I be an indexing set, and for each $i \in I$ suppose we have sets $\mathcal{X}_i$, and suppose that for each $i \in I$ we have an ordering $<_i$ on $\mathcal{X}_i$. Define $\mathcal{X} := \coprod_{i \in I} \mathcal{X}_i$. Then we have an embedding:

$$\mathbf{S}\mathcal{X}_i \overset{\iota_i}{\hookrightarrow} \mathbf{T}\mathcal{X}_i \hookrightarrow \mathbf{T}\mathcal{X} \tag{2}$$

given by writing any monomial $\underline{m}$ of length n in $\mathcal{X}_i$ as:

$$\underline{m} = x_1 x_2 x_3 \cdots x_n$$

where $x_i \in \mathcal{X}_i$ and $x_1 \leq_i x_2 \leq_i \cdots \leq_i x_n$, and we define:

$$\iota_i(\underline{m}) := x_1 x_2 x_3 \cdots x_n \in \mathbf{T}\mathcal{X}_i.$$

The map from $\mathbf{S}\mathcal{X}_i$ is then the R-linear extension of this map. By abuse of notation, we denote $\iota_i(\underline{m})$ simply by $\underline{m}$ and the composite map of (Eqn. **2**) also by ι_i.

Lemma 1.5. *Let $\overline{T}$ be a splay with coordinate algebras A_i over a field k. Suppose that each A_i has a fixed presentations:*

$$\mathcal{S}_i \hookrightarrow \mathbf{S}\mathcal{X}_i \twoheadrightarrow A_i$$

for some set $\mathcal{X}_i$ and ideal of relations $\mathcal{S}_i$. Suppose that for each i there is a map:

$$\mathbf{deg}_i : \mathcal{X}_i \longrightarrow \mathbb{Z}_{\geq 1}$$

and an ordering $<_i$ on $\mathcal{X}_i$ such that for some indexing set J_i there is a collection of polynomials $\{f_{i,j} \mid j \in J_i\}$ which forms a Gröbner basis for the ideal $\mathcal{S}_i$ with respect to the graded lexicographic ordering on $\mathbf{S}\mathcal{X}_i$ induced by $<_i$ and $\mathbf{deg}_i$. Then the splay algebra T has a k basis of elements of the form:

$$\underline{a}_1\underline{a}_2 \cdots \underline{a}_k$$

where for each $1 \leq j \leq k$ there is some $i_j \in I$ such that $i_j \neq i_{j+1}$ and such that $\underline{a}_j = a_{j,1} \cdots a_{j,n_j}$ are non-decreasing monomials of length $n_j \geq 1$ in $\mathcal{X}_{i_j}$ which are not leading monomial ideal of the ideal $\mathcal{S}_{i_j}$,

Proof. Adopting the notation of (Rmk. **1.4.2**), one may exhibit $\coprod_i A_i$ as:

$$T = \mathbf{T}\mathcal{X}/\mathcal{R}$$

where $\mathcal{R} = \sum_i \mathcal{R}_i$ and $\mathcal{R}_i$ is the ideal generated by $\mathcal{S}_i$ and the relations $s_{a,a'} := aa' - a'a$ for $a, a' \in A_i$.

The statement now reduces to showing that the set:

$$\bigcup_{i\in I}\{f_{i,j} \mid j \in J_i\} \cup \bigcup_{i\in I}\{s_{a,a'} \mid a >_i a' \in \mathcal{X}_i\}$$

is a Gröbner basis for the ideal $\mathcal{R}$ for some monomial ordering on $\mathbf{T}\mathcal{X}$. This follows readily from the graded lexicographic ordering given by $\mathbf{deg}$ and an ordering $\ll$ as follows. As $\mathcal{X}$ is the co-product in sets of the $\mathcal{X}_i$, the maps $\mathbf{deg}_i$ induce a map $\mathcal{X} \xrightarrow{\mathbf{deg}} \mathbb{Z}_{\geq 1}$. In concrete terms, if $\underline{m} = \underline{m_1} \cdots \underline{m_n}$ is a monomial in $\mathbf{T}\mathcal{X}$ with the $\underline{m_j}$ monomials in $\mathcal{X}_{i_j}$ for some $i_j \in I$, then we have:

$$\mathbf{deg}(\underline{m}) := \sum_{1\leq j\leq n} \mathbf{deg}_{i_j}(\underline{m_j}).$$

Let $x_i \in \mathcal{X}_i \subseteq \mathcal{X}$ and $x_{i'} \in \mathcal{X}_{i'} \subseteq \mathcal{X}$ we define:

$$x_i \ll x_{i'} \quad \text{if} \quad \begin{cases} i < i', \\ \text{OR} \\ i = i' \quad \text{and} \quad x_i <_i x_{i'}. \end{cases}$$

∎

Remark 1.6.

1.6.1. By the results in [11], we may easily extend (Lem. 1.5) to the case of a general ring R as long as we assume that the leading coefficients of the $f_{i,j}$ are units of R.

1.6.2. Under the assumptions of (Lem. 1.5) and the ordering $\ll$ introduced in its proof, we obtain an R-linear map:

$$T \xhookrightarrow{\iota_{\ll}} \mathbf{T}\mathcal{X}$$

such that $q \circ \iota_{\ll}$ is the identity map on T where q is the quotient map $\mathbf{T}\mathcal{X} \xrightarrow{q} T$. If $\mathcal{J}$ is an ideal of T, we denote by $\tilde{J}_{\ll}$ the ideal of $T\mathcal{X}$ generated by image of $\iota_{\ll}(\mathcal{J})$.

Suppose now that π is a Poisson structure on a splay $\overline{T}$. Then we obtain a Poisson structure on $\mathbf{T}\mathcal{X}$, denoted by $\tilde{\pi}$, by applying the bi-derivation law to the following map:

$$\begin{array}{ccccc} \mathcal{X} \times \mathcal{X} & \longrightarrow & A & \xhookrightarrow{\iota_{\ll}} & \mathbf{T}\mathcal{X} \\ (x_i, x_{i'}) & \longmapsto & \pi(x_i, x_i') & \longmapsto & \iota_{\ll} \circ \pi(x_i, x_{i'}). \end{array}$$

Lemma 1.7. *Let $\overline{T}$ be a splay over a field k with coordinate algebras A_i which satisfy the conditions of the (Lem. 1.5). Suppose that π Poisson structure on $\overline{T}$ Denote by $\mathcal{J}$ the T ideal:*

$$\mathcal{J} = (aa' - a'a - \pi(a, a'))\,.$$

Let $\tilde{\pi}_{\ll}$ and $\tilde{\mathcal{J}}_{\ll}$ be as in (Rmk. **1.6.2***) and $\mathcal{R}$ be as in the proof of (Lem. 1.5). Then:*

- *The Poisson structure $\tilde{\pi}_{\ll}$ descends to π.*
- *Suppose that $f, g \in \mathbf{T}\mathcal{X}$, then*

$$fg - gf \equiv \tilde{\pi}_{\ll}(f, g) \quad \Big(\mathbf{mod}\ \mathcal{R} + \tilde{\mathcal{J}}_{\ll}\Big).$$

- *π descends to the canonical Poisson structure on $U := T/\mathcal{J}$.*

Proof. The first statement is clear by the construction of $\tilde{\pi}_{\ll}$. The third statement follows readily from the second. For the second, by k-linearity, we may assume that f and g are monomials. In which case, we proceed by induction on the sum of their lengths $L := \mathbf{len}(f) + \mathbf{len}(g)$. For $L \leq 2$ it is trivial by definition of $\tilde{\pi}_{\ll}$. Suppose now that we have proved it for all $L' \leq L$. Let $x \in \mathcal{X}$ and consider $f' = fx$ where $\mathbf{len}(f) \leq L'$. Then, we have:

$$\begin{aligned} f'g - gf' &\equiv fxg - gfx \\ &\equiv (f\tilde{\pi}_{\ll}(x, g) + fgx) - (fgx + \tilde{\pi}_{\ll}(g, f)x) \\ &\equiv f\tilde{\pi}_{\ll}(x, g) + \tilde{\pi}_{\ll}(f, g)x \\ &\equiv \tilde{\pi}_{\ll}(fx, g) \equiv \tilde{\pi}_{\ll}(f', g). \end{aligned}$$

A similar statement for $\tilde{\pi}_{\ll}(g, f')$ gives the desired result. ∎

Remark 1.8. Let $\overline{T}$ (resp. $\overline{T}'$) be a splay with Poisson structure π (resp. π') and suppose that $\overline{T} \xrightarrow{\Phi} \overline{T}'$ is a Poisson morphism. Let U (resp. U) be as in (Lem. 1.7). Then it is clear that Φ descends to an algebra morphism $U \xrightarrow{\Phi} U'$. If A_i for $i \in I$ (resp. A'_j for $j \in J$) are the coordinate algebras of $\overline{T}$ (resp. $\overline{T}'$), then we have the following commutative diagram:

$$\begin{tikzcd} A_i \arrow[r, hook, "\iota_i"] \arrow[d, "\Phi_{j_i}"'] & T \arrow[r, two heads] & U \arrow[d, "\Phi"] \\ A'_{j_i} \arrow[r, hook, "\iota_{j_i}"] & T' \arrow[r, two heads] & U'. \end{tikzcd}$$

∎

We now prove a PBW-type theorem. In the case of a Lie algebra $\mathfrak{g}$ this reduces to the usual PBW theorem where one has chosen an ordered basis for the Lie algebra (c.f. [16], [15], [4]). Again, we may extend (Lem. 1.7) and (Rmk. 1.9) to the case of a general ground ring R as long as the leading coefficients of the $f_{i,j}$'s are units.

Theorem 1.9. *Suppose $\overline{T}$ is a splay over a field k with coordinate algebras A_i which satisfy the conditions of (Lem. 1.5). Suppose that π is a strongly filtered Poisson structure on $\overline{T}$, and let $\mathcal{J}$ be as in (Lem. 1.7). Then, fixing the choice of a total ordering $<$ on I, U is isomorphic as a k-module to $S := \bigotimes_i A_i$.*

Proof. It is now enough to show that the set:

$$\mathcal{Y} := \{f_{i,j} \mid i \in I, j \in J_i\} \cup \{g_{a,b} \mid a \gg b \in \mathcal{X}\}$$

is a Gröbner basis of $\mathcal{R} + \tilde{\mathcal{J}}_{\ll}$ with respect to **deg** and the ordering $\ll$ as in the proof of (Lem. 1.5).

We first show that $\mathcal{Y}$ generates $\mathcal{R}+\tilde{\mathcal{J}}_{\ll}$. By k-linearity of the Poisson bracket, and as $\mathcal{R} = \sum_i \mathcal{R}_i$ where $\mathcal{R}_i \subseteq \mathbf{T}\mathcal{X}_i$ has a Gröbner basis in $\mathbf{T}\mathcal{X}_i$ given by the $f_{i,j}$'s and the $g_{a,b} = ab - ba$ for $a >_i b \in \mathcal{X}_i$, it is enough to show that for all monomials m and n of $\mathbf{T}\mathcal{X}$, we may write $mn - nm - \tilde{\pi}(m,n) = \sum_r \mu_r g_{a_r,b_r} \eta_r$ for some $\mu_r, \eta_r \in \mathbf{T}\mathcal{X}$ and some $a_r \gg b_r \in \mathcal{X}$. The statement then follows from repeated application of the bi-derivation law. Indeed, writing $m = m'm''$, we have that:

$$\begin{aligned} & m'm''n - nm'm'' - \tilde{\pi}(m'm'', n) \\ &\quad = \quad m'm''n - nm'm'' - m'\tilde{\pi}(m'', n) - \tilde{\pi}(m', n)m'' \\ &\quad = \quad m'(m''n - \tilde{\pi}(m'', n)) - (nm' + \tilde{\pi}(m', n))m'' \\ &\quad = \quad m'(m''n - \tilde{\pi}(m'', n)) - m'nm'' + m'nm'' - (nm' + \tilde{\pi}(m', n))m'' \\ &\quad = \quad m'(m''n - nm'' - \tilde{\pi}(m'', n)) + (m'n - nm' - \tilde{\pi}(m', n))m'' \end{aligned}$$

which allows us to continually reduce the length of the monomials under consideration, so that we may assume that m and n both have length 1, i.e.

that they may be identified as elements of $\mathcal{X}$. Then we may write either $mn - nm - \tilde{\pi} = g_{m,n}$ if $m \gg n$ or otherwise $mn - nm - \tilde{\pi} = -g_{n,m}$.

We now show that this is indeed a Gröbner basis. Without loss of generality, we may assume that the leading coefficient $f_{i,j}$ is $\mathbf{LC}(f_{i,j}) = 1$. Recalling that the $f_{i,j}$ are polynomials in $\mathbf{S}\mathcal{X}_i \hookrightarrow \mathbf{T}\mathcal{X}_i$, we may write their leading monomials as:

$$\mathbf{LM}(f_{i,j}) =: \underline{x}_{i,j} = x_{i,j,1}x_{i,j,2}\cdots x_{i,j,n_j}$$

where the $x_{i,j,k} \in \mathcal{X}_i$ satisfy $x_{i,j,1} \leq_i x_{i,j,2} \leq_i \cdots \leq_i x_{i,j,n_j}$. For the polynomials $g_{y,x}$ with $y > x \in \mathcal{X}$, the strong filtration property on π ensures that $\mathbf{LM}(g_{y,x}) = yx$. Now it suffices to check the vanishing of the following "S-polynomials":

$$S_1 = g(x_k, x_{i,j,1})x_{i,j,2}\cdots x_{i,j,n_j} - x_k f_{i,j}$$

with $x_k \gg x_{i,j,1}$ in $\mathcal{X}$:

$$S_2 = x_{i,j,1}x_{i,j,2}\cdots x_{i,j,n_j-1}g(x_{i,j,n_j}, x_k) - f_{i,j}x_k$$

with $x_{i,j,n_j} \gg x_k$ in $\mathcal{X}$, and:

$$S_3(k,j,i) = g(x_k, x_j)x_i - x_k g(x_j, x_i)$$

with $x_k \gg x_j \gg x_i$ in $\mathcal{X}$.

Let us write $f_{i,j} = \underline{x}_{i,j} - h_{i,j}$. We have two possibilities in showing that S_1 vanishes. The first case is for $x_k \in \mathcal{X}_i$, in which we have for all $1 \leq r \leq n_j$:

$$g(x_k, x_{1,j,r}) = x_k x_{i,j,r} - x_{i,j,r}x_k$$

by internal symmetry. From this, the vanishing of S_1 follows easily. The second case is for $x_k \notin \mathcal{X}_i$, but as $x_k \gg x_{i,j,1}$, this means that $x_k \in \mathcal{X}_{i'}$ for some $i' > i$, so that $x_k \gg x_{i,j,1} \underline{\gg} g_{i,j_2}\cdots \underline{\gg} x_{i,j,n_j}$. We also note that as $\tilde{\pi}$ descends to π on A we necessarily have that for all $h \in \mathbf{T}\mathcal{X}$ we have that:

$$0 = \tilde{\pi}(f_{i,j}, h) = \tilde{\pi}(\underline{x}_{i,j} - h_{i,j}, h).$$

Now as $\tilde{\pi}$ is a bi-derivation, it follows that:

$$\sum_{1\leq r\leq n_j} x_{i,j,1}\cdots x_{i,j,r-1}\tilde{\pi}(x_{i,j,r}, h)x_{i,j,r+1}\cdots x_{i,j,n_j} = \tilde{\pi}(h_{i,j}, h)$$

With these observations, we may now compute S_1 as follows:

$$\begin{aligned}
S_1 &= -x_{i,j,1}x_k x_{i,j,2}\cdots x_{i,j,n_j} - \tilde{\pi}(x_k, x_{i,j,1})x_{i,j,2}\cdots x_{i,j,n_j} + x_k h_{i,j} \\
&= -x_{i,j,1}x_{i,j,2}x_k x_{i,j,3}\cdots x_{i,j,n_j} - x_{i,j,1}\tilde{\pi}(x_k, x_{i,j,2})x_{i,j,3}\cdots x_{i,j,n_j} \\
&\quad -\tilde{\pi}(x_k, x_{i,j,1})x_{i,j,2}\cdots x_{i,j,n_j} + x_k h_{i,j} \\
&\vdots \\
&= -x_{i,j,1}\cdots x_{i,j,n_j}x_k + x_k h_{i,j} \\
&\quad - \sum_{1\leq r\leq n_j} x_{i,j,1}\cdots x_{i,j,r-1}, \tilde{\pi}(x_k, x_{i,j,r})x_{i,j,r+1}\cdots x_{i,j,n_j} \\
&= -h_{i,j}x_k - \tilde{\pi}(x_k, \underline{x}_{i,j}) + x_k h_{i,j} \\
&= \tilde{\pi}(x_k, h_{i,j}) - \tilde{\pi}(x_k, \underline{x}_{i,j}) = 0
\end{aligned}$$

The vanishing of S_2 is similar.

The vanishing of S_3 proceeds similarly to the case of the Gröbner basis for the PBW theorem of a Lie algebra. Indeed, we have:

$$\begin{aligned}
S_3 &= -x_k x_i x_j - x_k\tilde{\pi}(x_j, x_i) + x_j x_k x_i + \tilde{\pi}(x, x_j)x_i \\
&= -x_i x_k x_j - \tilde{\pi}(x_k, x_i)x_j - x_k\tilde{\pi}(x_j, x_i) + x_j x_i x_k \\
&\quad +x_j\tilde{\pi}(x_k, x_i) + (x_k, x_j)x_i \\
&= -x_i x_j x_k - x_i\tilde{\pi}(x_k, x_j) - \tilde{\pi}(x_k, x_i)x_j - x_k(x_k, x_i) + x_i x_j x_k \\
&\quad +\tilde{\pi}(x_k, x_i)x_k + x_j\tilde{\pi}(x_k, x_i) + \tilde{\pi}(x_k, x_j)x_i \\
&= \tilde{\pi}(\tilde{\pi}(x, x_j), x_i) + \tilde{\pi}(x_j, \tilde{\pi}(x_k, x_i)) + \tilde{\pi}(\tilde{\pi}(x_j, x_i), x_k) \\
&= 0
\end{aligned}$$

where the last equality follows from skew-Symmetry along with the Jacobi identity. ∎

Remark 1.10.

1.10.1. We have injective maps $A_i \overset{\iota_i}{\hookrightarrow} T = \coprod_i A_i$, and it is clear that the composite maps $A_i \xrightarrow{q\circ\iota_i} U$ is injective, so that A_i is a k-sub-algebra of U.

1.10.2. Let $\mathfrak{g}$ be a Lie algebra over k and choose a basis $\mathfrak{g} = k\langle X_i\rangle_{i\in I}$. Take for A_i the free commutative algebra generated by x_i. Then $T = \mathbf{T}\mathfrak{g}$, and $U = \mathbf{T}\mathfrak{g}/\mathcal{J} = \mathfrak{U}\mathfrak{g}$, the universal enveloping algebra, and we may identify $S = \bigotimes A_i = \mathbf{S}\mathfrak{g}$. Choosing an ordering $<$ on I, we recover the usual PBW theorem.

1.10.3. More generally, let $\mathfrak{g}$ be a Lie algebra where we may write $\mathfrak{g} = \oplus_{i\in I}\mathfrak{h}_i$ where the $\mathfrak{h}_i$ are abelian Lie sub-algebras. Taking $A_i = \mathbf{S}\mathfrak{h}_i = \mathfrak{U}\mathfrak{h}_i$, then we have:

$$T = \mathbf{T}\mathfrak{g}/\mathcal{R}.$$

Here, $\mathcal{R}$ is the ideal $\mathcal{R} = \sum_i \mathcal{R}_i$ where $\mathcal{R}_i$ is the ideal generated by the elements $aa' - a'a$ for $a, a' \in A_i$. Then, the Lie algebra structure on $\mathfrak{g}$ induces a Poisson bracket on T, and as the $\mathfrak{h}_i$ are abelian sub-algebras, we see that the Poisson bracket has internal symmetry with respect to the A_i. We may identify $U = \mathfrak{U}\mathfrak{g}$, and we may view the $A_i = \mathbf{S}\mathfrak{h}_i$ as sub-algebras of U. Identifying $S = \bigotimes A_i = \mathbf{S}\mathfrak{g}$, and choosing an ordering on the indexing set I as well as an ordered basis for each of the $\mathfrak{h}_i$'s, we again recover the usual PBW theorem.

Definition 1.11. Let $\overline{T}$ be a splay with coordinate algebras A_i for $i \in I$ and suppose that $\overline{T}$ has a strongly filtered Poisson structure π. Denote the multiplication of $S := \otimes A_i$ be $\cdot$. Let $<$ be an ordering on I. Then we say that a map $T \xrightarrow{\Delta} T\otimes T$ is **strongly multiplicative** (with respect to $<$) if:

$$\Delta\pi = (\cdot\otimes\pi + \pi\otimes\cdot + \pi\otimes\pi)\circ(\mathbb{1}\otimes\tau\otimes\mathbb{1})\circ\Delta.$$

Here, $\mathbb{1}$ is the identity map, and $\cdot$ denotes, by abuse of notation, the map induced by on $\mathbf{T}\mathcal{X}$ by:

$$\mathcal{X} \times \mathcal{X} \xrightarrow{\cdot} \mathbf{S}\mathcal{X} \xhookrightarrow{\iota_{\ll}} T\mathcal{X}.$$

Remark 1.12.

1.12.1. Suppose that we are in the situation of (Def. 1.11). Then if $\eta, \zeta \in \mathcal{X}$ we have $\eta \cdot \zeta = \zeta \cdot \eta$. Moreover, if $\eta > \zeta$ we have by construction of T that:

$$\zeta\eta = \zeta \cdot \eta = \eta \cdot eta$$

1.12.2. Let us continue the example given by a Lie algebra $\mathfrak{g}$ with the choice of an ordered basis, where $\mathbf{T}\mathfrak{g} \xrightarrow{\Delta} \mathbf{T}\mathfrak{g} \otimes \mathbf{T}\mathfrak{g}$ is map induced by the diagonal map on $\mathfrak{g}$. Then, one has that the $\pi \otimes \pi$ term of the strong multiplicativity condition automatically vanishes when applied to elements of $\mathfrak{g} \subseteq T = \mathbf{T}\mathfrak{g}$ and, thus, what remains is the usual multiplicativity condition (c.f. [13]). Moreover, one can view the multiplicativity condition as exactly the condition that is needed to ensure that Δ descends to an algebra map $\mathfrak{U}\mathfrak{g} \xrightarrow{\Delta} \mathfrak{U}\mathfrak{g} \otimes \mathfrak{U}\mathfrak{g}$.

In view of the previous remark, we note the following:

Proposition 1.13. *Let $\overline{T}$ be a splay with Poisson structure π and coordinate algebras A_i for $i \in I$ satisfy the conditions of (Lem. 1.5). Suppose that for each i we have algebra maps $A_i \xrightarrow{\Delta_i} A_i \otimes A_i$, such that the induced map $T \xrightarrow{\Delta} T \otimes T$ is strongly multiplicative. Then Δ descends to an algebra map $U \xrightarrow{\Delta} U \otimes U$, where U is as in (Thm. 1.9).*

Proof. Let us fix an ordering on the index set I. In the proof of (Thm. 1.9) we realized U as the quotient of $\mathbf{T}\mathcal{X}$ by $\mathcal{R} + \tilde{\mathcal{J}}_{\ll}$ for which we have exhibited a Gröbner basis for $\mathcal{R} + \tilde{\mathcal{J}}_{\ll}$ given in terms of the $f_{i,j}$'s and $g_{y,x}$'s for $y \gg x \in \mathcal{X}$. Let us write π instead of $\tilde{\pi}$ by abuse of notation. Since Δ is induced from the Δ_i, it is enough to show that Δ vanishes on the $g_{y,x}$'s. Let us suppose that $y \in \mathcal{X}_j$ and $x \in \mathcal{X}_i$. If $i = j$, then $g_{y,x} = yx - xy$ and the assertion is trivial. If $i \neq j$, then $j > i$ as $y \gg x$. As Δ was induced from the Δ_i's, we may write:

$$\begin{aligned} \Delta(x) &= \Delta_i(x) = \sum_r \lambda_r x'_r \otimes x''_r \\ \Delta(y) &= \Delta_j(y) = \sum_s \mu_s y'_s \otimes y''_s. \end{aligned}$$

Here, the x'_r, x''_r are monomials in $\mathcal{X}_i$, and the y'_s, y''_s are monomials in $\mathcal{X}_j$. We

compute modulo the ideal $(\mathcal{R}+\tilde{\mathcal{J}}_{\ll})\otimes\mathbf{T}\mathcal{X}+\mathbf{T}\mathcal{X}\otimes(\mathcal{R}+\tilde{\mathcal{J}}_{\ll})$ in that:

$$\begin{aligned}\Delta(yx-xy) &\equiv \sum_{r,s}\lambda_r\mu_s\,(y'_s x'_r\otimes y''_s x''_r - x'_r y'_s\otimes x''_r y''_s)\\ &\equiv \sum_{r,s}\lambda_r\mu_s\,((x'_r y'_s+\pi(y'_s,x'_t))\otimes(x''_r y''_s+\pi(y''_s,x''_r)) - x'_r y'_s\otimes x''_r y''_s)\\ &\equiv \sum_{r,s}\lambda_r\mu_s\left(\begin{array}{l}\pi(y'_s,x'_r)\otimes x''_r y''_s + x'_r y'_s\otimes\pi(x''_r,y''_s)\\ +\pi(y'_s,x'_r)\otimes\pi(y''_s,x''_r)\end{array}\right)\\ &\equiv (\pi\otimes\cdot+\cdot\otimes\pi+\pi\otimes\pi)\circ(1\otimes\tau\otimes 1)\circ(\Delta\otimes\Delta)(y\otimes x)\\ &\equiv \Delta\circ\pi(y,x)\end{aligned}$$

which yields the desired result. ∎

2 Formal Groups and Distribution Algebras

We wish to apply the results of §1 in order to give a description of non-commutative formal groups over a field k of characteristic $p>0$. We establish some notation and conventions for general rings R. We say that $\mathbb{G}$ is a formal group (over R) if it is a group object in the category of smooth formal varieties over R. We denote its dimension by $n<\infty$.

We recall some basic results of such formal groups (c.f [2], [3], [7], [8], and [10]). Given a formal group $\mathbb{G}$ over R, one can look at the ring of formal functions $R[\![\mathbb{G}]\!]$ whose maximal ideal of functions we denote $\mathfrak{m}$. One can form the continuous linear dual of $R[\![\mathbb{G}]\!]$ with respect to the $\mathfrak{m}$-adic topology which we denote by $\mathbf{Dist}(\mathbb{G})$. We denote by $\langle\ \ ,\ \ \rangle$ the natural pairing between $\mathbf{Dist}(\mathbb{G})$ and $R[\![\mathbb{G}]\!]$. One knows that $\mathbf{Dist}(\mathbb{G})$ has an algebra structure given by dualizing the co-multiplication map m of $R[\![\mathbb{G}]\!]$. The non-commutativity of $\mathbb{G}$ is reflected equally in the non-commutativity of the algebra structure of $\mathbf{Dist}(\mathbb{G})$ and the non-co-commutativity of the co-multiplication m. One knows further that one may dualize the multiplication of $R[\![\mathbb{G}]\!]$ to give a co-multiplication map Δ on $\mathbf{Dist}(\mathbb{G})$. The algebra and co-algebra structure of $\mathbf{Dist}(\mathbb{G})$, and also those of $R[\![\mathbb{G}]\!]$, are compatible so that we in fact have a bi-algebra structure. The $\mathfrak{m}$-adic topology on $R[\![\mathbb{G}]\!]$ yields a filtration of $\mathbf{Dist}(\mathbb{G})$ by sub-co-algebras:

$$\begin{aligned}\mathbf{Dist}(\mathbb{G}) &= \bigcup_{n\geq 0}\mathbf{Dist}^{(n)}(\mathbb{G})\\ \mathbf{Dist}^{(n)}(\mathbb{G}) &:= R\oplus\mathbf{Dist}_+^{(n)}(\mathbb{G})\\ \mathbf{Dist}_+^{(n)}(\mathbb{G}) &:= \mathbf{Hom}_R(\mathfrak{m}/\mathfrak{m}^{n+1},R).\end{aligned}$$

A choice of coordinate system $\underline{x}=\{x_1,\cdots,x_n\}$ on $\mathbb{G}$ gives an identification $R[\![\mathbb{G}]\!]=R[\![x_1,\cdots,x_n]\!]$. If one chooses an ordering on the x_i's, say $x_1<\cdots<x_n$, then then one is working in the category of formal group laws. Any choice of a coordinate system, without any specified ordering, gives the choice of an additional structure on $\mathbf{Dist}(\mathbb{G})$, namely that of a **DVPS-bi-algebra**. By this

we mean there is a choice of a basis $\delta_{\underline{x}^J}$ of $\mathbf{Dist}(\mathbb{G})$ where $J = (J_1, \cdots, J_n) \in \mathbb{Z}^n_{\geq 0}$ is a multi-index such that the co-multiplication law satisfies:

$$\Delta(\delta_{\underline{x}^J}) = \sum_{A+B=J} \delta_{\underline{x}^A} \otimes \delta_{\underline{x}^B}.$$

The specific basis that the choice of $\underline{x}$ is the basis given by the maps:

$$\delta_{\underline{x}^J}(\underline{x}^K) := \begin{cases} 1 & J = K \\ 0 & J \neq K \end{cases}$$

where for the multi-index K we have defined $\underline{x}^K := x_1^{K_1} \cdots x_n^{K_n} \in R[\![\mathbb{G}]\!]$. We will call such a basis an **additive basis**. The existence of the additive basis means that one can recognize the dual of $\mathbf{Dist}(\mathbb{G})$ with multiplication given by dualizing Δ as a ring of power series. There is a final structure on $\mathbf{Dist}(\mathbb{G})$ which is the anti-pode or inverse $\mathbf{Dist}(\mathbb{G}) \xrightarrow{\mathbf{inv}} \mathbf{Dist}(\mathbb{G})$ which gives $\mathbf{Dist}(\mathbb{G})$ the structure of a Hopf algebra. The existence and uniqueness is automatic in the situations we will consider below.

Example 2.1.

2.1.1. Let $\mathbf{G}$ be a smooth (analytic, algebraic,..) group defined over R. Then we denote the completion of its ring of functions at the identity by $R[\![\mathbf{G}]\!]$, which we identify as the ring of formal functions of a formal group. Denote by $\mathfrak{g}$ the Lie algebra of $\mathbf{G}$. If R is a $\mathbb{Q}$-algebra, then $\mathfrak{U}\mathfrak{g}$ is canonically isomorphic to $\mathbf{Dist}(\mathbf{G})$.

2.1.2. If k is a field of characteristic $p > 0$, or more generally R is an $\mathbb{F}_p$-algebra, then one has that $\mathbf{Dist}(\mathbb{G})$ is filtered by sub-bi-algebras:

$$\mathbf{Dist}(\mathbb{G}) = \bigcup_{r \geq 0} \mathbf{Dist}^{(p^r-1)}(\mathbb{G}).$$

Let us now assume that $R = k$ is a field of characteristic $p > 0$. In order to make use of the results of §1, we will need to make use of a second basis:

Lemma/Definition 2.2. Let $\mathbb{G}$ be a formal group over a field k of characteristic $p > 0$. Let $\underline{x}_< = \{x_1 < \cdots < x_n\}$ be the choice of an ordered coordinate system for $\mathbb{G}$. Then $\mathbf{Dist}(\mathbb{G})$ has a k-basis, the **multiplicative basis**, given for multi-indices J as the ordered products:

$$\delta^J_{\underline{x}_<} = \delta^{J_1}_{x_1} \cdots \delta^{J_n}_{x_n}$$

where:

$$\delta^{J_j}_{x_j} := \delta^{J_{j,0}}_{x_j^{p^0}} \cdots \delta^{J_{j,r}}_{x_j^{p^r}}.$$

Here, the $\delta_{x_j^{p^s}}$'s are as in the additive basis above, and the $J_{j,t}$'s are given by the p-adic expansions of the J_j's:

$$J_j = J_{j,0}p^0 + J_{j,1}p^1 + \cdots + J_{j,r}p^r.$$

When the choice of an ordered coordinate system $\underline{x}_<$ is understood, we will drop the subscripts and simply write $\delta^{\underline{J}} := \delta^{J}_{\underline{x}_<}$.

Remark 2.3.

2.3.1. The fact that this is indeed a basis is in [3]. Defining:

$$\mathcal{X} := \{\delta^{p^r}_{x_j} \mid 1 \leq j \leq n \ \ r \geq 0\}$$

one has at once an induced R-linear map $\iota_{\underline{x}_<}$ given by:

$$\begin{array}{ccc} \mathbf{Dist}(\mathbb{G}) & \xhookrightarrow{\iota_{\underline{x}_<}} & \mathbf{T}\mathcal{X} \\ \delta^{\underline{I}} & \longmapsto & \delta^{\underline{I}}. \end{array}$$

We define for $\eta, \zeta \in \mathcal{X} \subseteq \mathbf{Dist}(\mathbb{G})$:

$$\begin{array}{rcl} \tilde{\pi}_{\underline{x}_<}(\eta,\zeta) & := & \iota_{\underline{x}_<}(\eta\zeta - \zeta\eta) = \iota_{\underline{x}_<} \circ \pi_{\mathbf{c}}(\eta,\zeta) \\ g_{\underline{x}_<,\eta,\zeta} & := & \iota_{\underline{x}_<}(\eta)\iota_{\underline{x}_<}(\zeta) - \iota_{\underline{x}_<}(\zeta)\iota_{\underline{x}_<}(\eta) - \tilde{\pi}_{\underline{x}_<}(\eta,\zeta) \\ F_{\underline{x}_<}(\eta) & := & \iota_{\underline{x}_<}(\eta^p) \\ f_{\underline{x}_<,\eta} & := & \iota_{\underline{x}_<}(\eta)^p - F_{\underline{x}_<}(\eta). \end{array}$$

The extension of $\tilde{\pi}_{\underline{x}_<}$ by the bi-derivation law to $\mathbf{T}\mathcal{X}$ will also be denoted by $\tilde{\pi}_{\underline{x}_<}$. If we take the choice $\underline{x}_<$ as understood we shall write $\tilde{\pi} = \tilde{\pi}_{\underline{x}_<}$. We may also identify $\mathcal{X} \subseteq \mathbf{Dist}(\mathbb{G})$ and $\mathbf{Dist}(\mathbb{G})$ with their images in $\mathbf{T}\mathcal{X}$ via $\iota_{\underline{x}_<}$. Thus we may write more succinctly:

$$\begin{array}{rcl} g_{\eta,\zeta} & = & \eta\zeta - \zeta\eta - \tilde{\pi}(\eta,\zeta) \\ f_\eta & = & \eta^p - F(\eta) \end{array}$$

2.3.2. We have the following important filtration properties of the $F(\eta)$ and $\pi_{\mathbf{c}}(\eta,\zeta)$:

$$\begin{array}{rcl} F(\delta_{x_i^{p^r}}) & \in & \mathbf{Dist}^{(p^{r+1}-1)}(\mathbb{G}) \\ \pi(\delta_{x_j^{p^s}}, \delta_{x_i^{p^r}}) & \in & \mathbf{Dist}^{(p^r+p^s-1)}(\mathbb{G}). \end{array}$$

2.3.3. There is an analogue to the multiplicative basis in the characteristic zero situation of $\mathfrak{g}$ as above. For simplicity, let us assume that $\mathfrak{g}$ is abelian and arises as the Lie algebra of the formal group $\mathbb{G} = \mathbb{G}_a^n$. Let us choose a coordinate y_i for each of the $\mathbb{G}_a$'s such that we may write the co-multiplication map of $\mathbb{G}$ as:

$$m(y_i) = y_i \otimes 1 + 1 \otimes y_i.$$

Let us denote the corresponding basis of $\mathfrak{g}$ by $g = \langle Y_i \rangle_{1 \leq i \leq n}$ where $Y_i(y_j) = \delta_{i,j}$ is the Kronecker delta product. Then, in this setting the, "multiplicative basis" of $\mathfrak{U}\mathfrak{g} = \mathbf{S}\mathfrak{g} = \mathbf{Dist}(\mathbb{G})$ is given by the elements:

$$Y^{\underline{I}} = Y_1^{I_1} \cdots Y_n^{I_n}.$$

We may compute:

$$\begin{aligned}
\delta_{y_i}\delta_{y_i^k} &= \sum_J \langle m(\underline{y}^J), \delta_{y_i} \otimes \delta_{y_i}^k \rangle \delta_{\underline{y}^J} \\
&= \sum_j \langle (y_i \otimes 1 + 1 \otimes y_i)^j, \delta_{y_i} \otimes \delta_{y_i}^k \rangle \delta_{y_i^j} \\
&= (k+1)\delta_{y_i^{k+1}}
\end{aligned}$$

so that identifying $Y_i = \delta_{y_i}$ the additive and multiplicative basis are related by:

$$Y^{\underline{I}} = \delta_{\underline{y}}^{\underline{I}} = I!\delta_{\underline{y}^I}$$

where $I! := I_1! \cdots I_n!$.

We may have also chosen the coordinates $x_i = \exp(y_i) - 1$ for $\mathbb{G}$, so that:

$$m(x_i) = x_i \otimes 1 + 1 \otimes x_i + x_i \otimes x_i$$

which is the group law for $\mathbf{G}_m$ the multiplicative group as well as its formal completion $\mathbb{G}_m$. Then as we have that:

$$\delta_{x_i} \cdot \delta_{x_i^k} = (k+1)\delta_{x_i^{k+1}} + k\delta_{x_i^k}$$

one sees that the change of basis between the multiplicative and additive bases can be expressed in terms of the Sterling numbers of the first kind:

$$\delta_{x_i^n} = \frac{1}{n!}\delta_{x_i}(\delta_{x_i} - 1)(\delta_{x_i} - 2) \cdots (\delta_{x_i} - (n-1)).$$

2.3.4. The impossibility of inverting $I!$ and the fact that $\mathbb{G}_a$ and $\mathbb{G}_m$ are not isomorphic in positive characteristic is the essential source of all the complications. Let us define for $n \in \mathbb{Z} \geq 0$:

$$n!_p := n_0!n_1! \cdots n_r!$$

where $n = n_0 + n_1 p + \cdots n_r p^r$ is the p-adic expansion of n. We make a similar definition for a multi-index I. Then one know that for a formal group $\mathbb{G}$ we have:

$$\delta_{\underline{x}^I} = \frac{1}{I!_p}\delta_{\underline{x}_<}^{\underline{I}} + \quad \text{Lower Order Terms}$$

In the case of $\mathbb{G} = \mathbb{G}_a$ with coordinate y as above, we have, in fact, that:

$$\delta_{y^n} = \frac{1}{n!_p}\delta_{\underline{y}}^{\underline{n}}.$$

For $\mathbb{G} = \mathbb{G}_m$, with coordinate x as above, we have "Sterling numbers modulo p". Indeed, we may compute:

$$\delta_{x^{p^r}}\delta_{x^m} = (m_r + 1)\delta_{x^{m+1}} + m_r\delta_{x^m}$$

where $m = m_0 + \cdots + m_r p^r$ is the p-adic expansion of m and we assume $m_r < p-1$. From this it follows that we have:

$$\delta_{x^m} = \frac{1}{m!_p} \prod_{t=0}^{r} \delta_{x^{p^t}}(\delta_{x^{p^t}} - 1)(\delta_{x^{p^t}} - 2)\cdots(\delta_{x^{p^t}} - (m_t - 1)).$$

Proposition 2.4. *Let $\mathbb{G}$ be a formal group over a field k of characteristic $p > 0$. Let $\underline{x}_<$ be the choice of an ordered coordinate system. Let us adopt the notation as in (Rmk.* **2.3.1***). Define $\mathbf{deg}(\delta_{x_i}^{p^r}) := p^r$ and take:*

$$\delta_{x_j^{p^s}} > \delta_{x_i^{p^r}}$$

if $j > i$ or if $j = i$ and $s > r$. Let $\mathcal{I}$ be the two-sided ideal of $\mathbf{T}\mathcal{X}$:

$$\begin{aligned} \mathcal{I} &= \mathcal{R} + \tilde{\mathcal{J}}_{\ll} \\ \mathcal{R} &= (f_\eta \mid \eta \in \mathcal{X}) \\ \tilde{\mathcal{J}}_{\ll} &= (g_{\eta,\zeta} \mid \eta > \zeta \in \mathcal{X}) \,. \end{aligned}$$

Imposing the graded lexicographic ordering on $\mathbf{T}\mathcal{X}$, the set:

$$\{f_\eta, g_{\eta,\zeta} \mid \eta > \zeta \in \mathcal{X}\}$$

is a Gröbner basis for the ideal $\mathcal{I}$. In particular, $\mathbf{Dist}(\mathbb{G})$ *is isomorphic as an algebra to $\mathbf{T}\mathcal{X}/\mathcal{I}$.*

Proof. Due to the monomial ordering indicated and by (Rmk. **2.3.2**) we see that we having the following leading monomials:

$$\begin{aligned} \mathbf{LM}(f_\eta) &= \eta^p \\ \mathbf{LM}(g_{\eta,\zeta}) &= \eta\zeta. \end{aligned}$$

The result follows as otherwise we would not have the existence of a multiplicative basis. ∎

3 Geometric Formal Groups

Although (Prop. 2.4) gives some reasonable control on the algebra structure of $\mathbf{Dist}(\mathbb{G})$, some difficulties remain in trying to characterize formal groups in such a manner. First, our construction depended on the choice of an ordered coordinate system. Second, we have yet to address the DVPS-co-algebra structure and its compatibility with the algebra structure. For this, we will need to make an assumption on $\mathbb{G}$, namely that it is "geometric" which loosely means that one can choose a coordinate system of commutative formal subgroups. One may think of this as specifying a "local abelian structure". In the case where the ground ring is a field of characteristic zero, the fact that any commutative formal group is isomorphic to a product of $\mathbb{G}_a$'s means that there is essentially one local

abelian structure. However, in the positive characteristic situation, there are non-isomorphic (commutative) one-dimensional formal groups. Moreover, there are commutative formal groups which are not the products of one-dimensional formal groups, even over an algebraically closed field.

Definition 3.1.

3.1.1. Let $\mathbb{G}$ be a formal group over R. We say that $\mathbb{G}$ is **geometric** if there are commutative formal subgroups $\mathbb{H}_i$ for i in some indexing set I such that there is an isomorphism as between $\mathbb{G}$ and the product of formal varieties $\prod_i \mathbb{H}_i$. We denote a geometric formal group by the data $\{\mathbb{H}_i \xhookrightarrow{\iota_i} \mathbb{G}\}_{i \in I}$, or more briefly $\{\mathbb{H}_i, \mathbb{G}\}_{i \in I}$.

3.1.2. A **morphism** of geometric formal groups:

$$\{\mathbb{H}_i, \mathbb{G}\}_{i \in I} \xrightarrow{\Phi} \{\mathbb{H}'_j, \mathbb{G}'\}_{j \in J}$$

is a morphism of formal groups $\mathbb{G} \xrightarrow{\Phi} \mathbb{G}'$ such that for each $i \in I$ there is some $j_i \in J$ and some morphism of commutative formal groups $\mathbb{H}_i \xrightarrow{\Phi_{j_i}} \mathbb{H}'_j$ such that Φ is the morphism induced by the UMP of products of formal varieties as in the following diagram:

$$\begin{tikzcd}
\mathbb{H}_i \arrow[r, hook, "\iota_i"] \arrow[d, "\Phi_{j_i}"'] & \mathbb{G} \arrow[ddl, dashed, "\exists!"] \\
\mathbb{H}'_{j_i} \arrow[d, "\iota_{j_i}"'] & \\
\mathbb{G}' &
\end{tikzcd}$$

3.1.3. Let $\{\mathbb{H}_i, \mathbb{G}\}_{i \in I}$ be a geometric formal group over R. We define the **underlying commutative geometric group** as:

$$\mathbf{Comm}(\mathbb{H}_i, \mathbb{G}) := \{\mathbb{H}_i, \mathbb{H}\}$$

where $\mathbb{H} := \prod_i \mathbb{H}_i$ is the **underlying commutative group** and the product is taken in the category of commutative formal groups.

3.1.4. Let $\{\mathbb{H}_i, \mathbb{G}\}_{i \in I}$ be a geometric formal group over R. Let $d_i := \dim \mathbb{H}_i$ and denote by abuse of notation $I = |I|$, so that we may identify $I = \{1, \cdots |I|\}$. Suppose that for each $i \in I$, we have chosen an ordered coordinate system $\underline{x}_{i,<_i} = \{x_{i,1} <_i \cdots <_i x_{i,d_i}\}$. Let us suppose moreover that we have chosen an ordering $<$ on I. We will call the resulting ordered coordinate system:

$$\begin{aligned}
\underline{x}_{\ll} \quad &:= \quad \{x_{1,1} \ll \cdots \ll x_{1,d_1} \ll x_{2,1} \ll \cdots x_{I,1} \ll \cdots \ll x_{I,d_I}\} \\
&=: \quad \{x_1 \ll \cdots \ll x_n\}
\end{aligned}$$

a **geometric coordinate system**. We will call the data:

$$\{\underline{x}_{\ll}, \mathbb{H}_i, \mathbb{G}\}_{i \in I}$$

a **geometric formal group law**.

Remark 3.2.

3.2.1. Suppose that $\{\mathbb{H}_i, \mathbb{G}\}$ is a geometric formal group. Then, each of the $\mathbf{Dist}(\mathbb{H}_i)$ carries its own co-multiplication map, say Δ_i, as well as $\mathbb{H}$ with its co-multiplication Δ. However, as we have assumed an isomorphism of formal varieties $\mathbb{G} = \prod_i \mathbb{H}_i$, we have an isomorphism of co-algebras $\mathbf{Dist}(\mathbb{G}_i) = \prod_i \mathbf{Dist}(\mathbb{H}_i)$ under which we have $\Delta = \prod_i \Delta_i$.

3.2.2. One sees that the $\mathbf{Dist}(\mathbb{H}_i)$ are sub-bi-algebras of $\mathbf{Dist}(\mathbb{G})$ which generate $\mathbf{Dist}(\mathbb{G})$ as an algebra. The choice of the commutative formal subgroups $\mathbb{H}_i$ are not unique and, in particular, in positive characteristic may not be isomorphic.

Example 3.3.

3.3.1. If $\mathbb{G}$ is a commutative formal group, then it is geometric with $I = \{1\}$ and $\mathbb{H}_1 = \mathbb{G}$.

3.3.2. Suppose that $\mathbf{G}$ is a connected and simply-connected Lie group over a field k. If $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$ as in (Ex. **2.1.1**), then for any choice of basis $k\langle X_i \rangle$ we can exhibit the formal completion of $\mathbf{G}$ as a geometric formal group via the formal completions of the Lie subgroups $\mathbf{H}_i = \mathbf{exp}(tX_i)$. We have that the $\mathbf{Dist}(\mathbf{H}_i)$ are exactly the sub-bi-algebras $\mathbf{S}\{X_i\} = \mathbf{Dist}(\mathbf{H}_i)$ of $\mathfrak{U}\mathfrak{g} = \mathbf{Dist}(\mathbf{G})$. In particular, in characteristic zero all formal groups are geometric.

3.3.3. If k is an algebraically closed field of positive characteristic p, then the formal completion of any reductive algebraic group is geometric with formal subgroups either $\mathbb{G}_a$ or $\mathbb{G}_m$.

3.3.4. In general, we may have more than one type of a multiplicative coordinate. For example, consider in the case of $\mathbb{SL}_2$, we have coordinates (w, x, y) of type $(\mathbb{G}_a, \mathbb{G}_m, \mathbb{G}_a)$ given by:

$$\begin{pmatrix} 1+x & y \\ w & 1+z \end{pmatrix}$$

where $1+z = \frac{1+yw}{1+x}$. We also have coordinates (s, t, u) of type $(\mathbb{G}_a, \mathbb{G}_a, \mathbb{G}_a)$ given by:

$$\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Remark 3.4. Let $R = k$ be a field of positive characteristic. Let $\{\underline{x}_{\ll}, \mathbb{H}_i, \mathbb{G}\}_{i\in I}$ be a geometric formal group law. We may apply (Prop. 2.4) to each of the commutative formal subgroups $\mathbb{H}_i$. If we define for $i \in I$:

$$\mathcal{X}_i := \{\delta_{x_{i,k}^{p^r}} \mid 1 \leq k \leq d_i \quad r \geq 0\}$$

then $\mathcal{X} = \coprod_i \mathcal{X}_i$. Fixing an ordering $<$ on I, let us write $\iota_{\underline{x}_{\ll}} := \iota_{\ll}$ and $\tilde{\pi}_{\underline{x}_{<}} := \tilde{\pi}_{\ll}$ if there is no danger of confusion. As the $\mathbb{H}_i$'s are commutative formal subgroups, we have that:

$$F(\mathcal{X}_i) \subseteq \mathbf{Dist}(\mathbb{H}_i)$$

in other words:

$$f_\eta \subseteq \mathbf{T}\mathcal{X}_i \subseteq \mathbf{T}\mathcal{X} \qquad \forall \eta \in \mathcal{X}_i.$$

Further, we have that for $\eta, \zeta \in \mathcal{X}_i$ that:

$$g_{\eta,\zeta} = \eta\zeta - \zeta\eta \in \mathbf{T}\mathcal{X}_i \subseteq \mathbf{T}\mathcal{X}$$

∎

Lemma 3.5. *Let us adopt the notation and assumption of (Rmk. 3.4). Then $\tilde{\pi}_{\ll}$ descends to a strongly filtered, strongly multiplicative Poisson structure $\pi_{\ll}$ on the splay $\overline{T}$ of the $\mathbf{Dist}(\mathbb{H}_i)$.*

Proof. By the presentation of $\mathbf{Dist}(\mathbb{G})$ as given in (Prop. 2.4), and since we have $\tilde{\pi}_{\ll}(\mathcal{X}_i, \mathcal{X}_i) = 0$, we see that $\tilde{\pi}_{\ll}$ descends as we may present T as:

$$T = \mathbf{T}\mathcal{X} / \sum_i (f_\eta, g_{\eta,\zeta} \mid \eta, \zeta \in \mathcal{X}_i).$$

The internal symmetry follows. The strong filtration property follows by (Rmk. **2.3.2**). As $\mathbf{Dist}(\mathbb{G})$ is in fact a bi-algebra, one has by (Rmk. 3.2) the strong multiplicativity.

∎

Proposition 3.6. *Let k be a field of characteristic $p > 0$. Let $\{\mathbb{H}_i, \mathbb{G}\}_{i\in I}$ be a geometric formal group over k. Suppose that $\underline{x}_{\ll}$ (resp. $\underline{x'}_{\ll'}$) is a geometric coordinate systems on $\mathbb{G}$ induced by the choice of an ordering $<$ (resp. $<'$) on I and ordered coordinate systems $\underline{x}_{i,<_i}$ (resp. $\underline{x}'_{i',<'_i}$) on the $\mathbb{H}_i$. Let π (resp. π') be the resulting Poisson structure on T as in (Lem. 3.5). Then π and π' are equivalent. In particular if $<=<'$, then $\Phi = \mathbb{1}$.*
If $\{\mathbb{H}_i, \mathbb{G}\}_{i\in I} \xrightarrow{\Phi} \{\mathbb{H}'_j, \mathbb{G}'\}_{j\in J}$ is a morphism of geometric formal groups, then there is an induced Poisson morphism $\overline{T} \xrightarrow{\Phi} \overline{T}'$.

Proof. Let us first suppose that we are working with a fixed ordering $<=<'$ on I. Denote by Φ_i the isomorphisms:

$$\mathbf{Dist}(\mathbb{H}_i) = k\langle\delta_{\underline{x}_i}^{J_i}\rangle \xrightarrow{\Phi_i} k\langle\delta_{\underline{x}'_i}^{J'_i}\rangle = \mathbf{Dist}(\mathbb{H}_i)$$

given by the choices of the two multiplicative bases for the $\mathbf{Dist}(\mathbb{H}_i)$. Adopting the notation of (Lem. 2.4) for each of the coordinate systems, we denote the respective generating sets by $\mathcal{X} = \coprod \mathcal{X}_i$ and $\mathcal{X}' = \coprod \mathcal{X}_i'$. The choice of the different coordinate systems gives two different presentations of T as:

$$\begin{aligned} T &= \mathbf{T}\mathcal{X}/\mathcal{R} + \sum_i (\eta\zeta - \zeta\eta \mid \eta, \zeta \in \mathcal{X}_i) \\ T &= \mathbf{T}\mathcal{X}'/\mathcal{R}' + \sum_i (\eta'\zeta' - \zeta'\eta' \mid \eta', \zeta' \in \mathcal{X}_i) \end{aligned}$$

The Φ_i induce an isomorphism between these two presentations. The Poisson structures under consideration π and π' are induced respectively from:

$$\begin{aligned} \tilde{\pi} &:= \iota \circ \pi_{\mathbf{c}} \text{ for} & \mathbf{Dist}(\mathbb{G}) &\xhookrightarrow{\iota} \mathbf{T}\mathcal{X} \\ \tilde{\pi} &:= \iota' \circ \pi_{\mathbf{c}} \text{ for} & \mathbf{Dist}(\mathbb{G}) &\xhookrightarrow{\iota'} \mathbf{T}\mathcal{X}'. \end{aligned}$$

Our aim is to show that $\Phi \circ \pi = \pi' \circ (\Phi \otimes \Phi)$. First we suppose that we have fixed coordinate systems $\underline{x_i} = \underline{x_i}'$ on each of the $\mathbb{H}_i$ so that we have only chosen to change the ordering of $<_i$ to $<_i'$. As the induced multiplicative expansions of $\mathbf{Dist}(\mathbb{G})$ are equal up to recordings within each of the $\mathcal{X}_i$, we see that the induced map Φ simply induces a change of ordering of the monomials in the $\mathcal{X}_i$. The result follows in this setting.

Let us drop the assumption that $\underline{x}_i = \underline{x}_i'$. By what we have just said, after a possible reordering of the coordinates for each of the $\mathbf{Dist}(\mathbb{H}_i)$, we may assume that that the orderings $<_i$ and $<_i'$ are the same. Recalling that π and π' are defined in terms of $\pi_{\mathbf{c}}$, where we have:

$$\begin{aligned} \Phi \circ \pi_{\mathbf{c}}(\eta, \zeta) &= \Phi(\eta)\Phi(\zeta) - \Phi(\eta)\Phi(\zeta) \\ &= \pi_{\mathbf{c}}(\Phi(\eta), \Phi(\zeta)) \end{aligned}$$

the result now follows by (Lem. 1.7), by noticing that the multiplicative expansion of $\mathbf{Dist}(\mathbb{G})$ in terms of the coordinates $\underline{x}'$ can be obtained from multiplicative expansion of $\underline{x}'$ by applying Φ and the relations $\mathcal{R}'$.

Now we only need to illustrate the effect of a change of ordering $<$ on I to $<'$. Let us suppose that the ordering $<$ is the usual ordering $1 < 2 < \cdots < |I|$. Then, we may view any other ordering $<'$ on I as being given by $<' = <^{\sigma}$ for some permutation $\sigma \in \Sigma_I$. As the permutations are generated by the transpositions, we only need consider the case $\sigma = (i \quad i+1)$. In this case, we consider the automorphism Φ of T induced by $\mathbf{Dist}(\mathbb{H}_j) \xrightarrow{\Phi_j} \mathbf{Dist}(\mathbb{H}_j)$ where:

$$\Phi_j = \begin{cases} \mathbf{inv}_j & j \in \{i, i+1\} \\ \mathbb{1} & \text{otherwise.} \end{cases}$$

Here, $\mathbf{inv}_j$ is the inverse of $\mathbb{H}_j$. The fact that this gives an equivalence of Poisson structures follows from the fact that $\mathbf{inv}$ is an anti-algebra homomorphism so

that for $\eta, \zeta \in \mathcal{X}_i \cup \mathcal{X}_{i+1}$:

$$\begin{aligned} \mathbf{inv} \circ \pi_{\mathbf{c}}(\eta, \zeta) &= \mathbf{inv}(\eta\zeta - \zeta\eta) \\ &= \mathbf{inv}(\eta)\,\mathbf{inv}(\zeta) - \mathbf{inv}(\zeta)\,\mathbf{inv}(\eta) \\ &= \pi_{\mathbf{c}}(\mathbf{inv}(\zeta), \mathbf{inv}(\eta)). \end{aligned}$$

To obtain the morphism $\overline{T} \xrightarrow{\Phi} \overline{T}'$, we first choose an ordering $<'$ on the set J. Then we may choose any ordering $<$ on I such that if $i_1 < i_2$ then $j_{i_1} <' j_{i_2}$ where the j_i's are as in the definition of a geometric morphism. We note that such an ordering exists: We fix a choice of such j_i's and define $I_j := \{i \in I \mid j_i = j\}$. Then as we have $I = \coprod_{j \in J} I_j$, we choose any ordering $<_j$ on the I_j and define the ordering $<$ on I as $i_1 < i_2$ if either $j_{i_1} < j_{i_2}$ or $j := j_{i_1} = j_{i_2}$ and $i_1 <_j i_2$. The result follows readily. ∎

Theorem 3.7. *Let $\{\mathbb{H}_i, \mathbb{H}\}$ be a commutative geometric formal group. Suppose that π is a strongly filtered, strongly multiplicative Poisson structure on the splay $\overline{T}$ of the $\mathbf{Dist}(\mathbb{H}_i)$. Let $\mathcal{J}$ be the ideal in the splay algebra T:*

$$\mathcal{J} = (fg - gf - \pi(f, g) \mid f, g \in T).$$

Then the quotient:

$$U := T/\mathcal{J}$$

is the distribution algebra of a formal group, $\mathbb{G}$. Moreover, $\{\mathbb{H}_i, \mathbb{G}\}$ is a geometric formal group.
Suppose that $\overline{T}'$ is the splay of a commutative geometric formal group $\{\mathbb{H}'_i, \mathbb{H}'\}$ with a strongly filtered, strongly multiplicative Poisson structure π'. Suppose that $\overline{T} \xrightarrow{\Phi} \overline{T}$ is a Poisson morphism, then there is morphism of geometric formal groups $\{\mathbb{H}_i, \mathbb{G}\} \xrightarrow{\Phi} \{\mathbb{H}'_i, \mathbb{G}'\}$.

Proof. Let us choose an ordering $<$ on I. By the work that we have done in §1 we have a bi-algebra structure on U. Thus, we only need to show that there is a DVPS algebra structure on U. Let us write:

$$\{x_1 \ll \cdots \ll x_n\} := \{x_{1,1} \ll \cdots \ll x_{|I|, d_{|I|}}\}.$$

For a multi-index J, we define the following element of U:

$$\delta_{x^J} := \delta_{x_1^{J_1}} \cdots \delta_{x_n^{J_n}}.$$

If $1 \leq j \leq n$, let $i_j \in I$ denote the index such that x_j is a coordinate of $\mathbb{H}_{i_j}$.

Then, as we have taken care of the ordering, and by (Rmk. 3.2), we have:

$$\begin{aligned}
\Delta(\delta_{x^J}) &= \Delta_{i_1}(\delta_{x_1^{J_1}})\cdots\Delta_{i_n}(\delta_{x_n^{J_n}}) \\
&= \left(\sum_{a_1+b_1=J_1}\delta_{x_1^{a_1}}\otimes\delta_{x_1^{b_1}}\right)\cdot\left(\sum_{a_n+b_n=J_n}\delta_{x_n^{a_n}}\otimes\delta_{x_n^{b_n}}\right) \\
&= \sum_{a_j+b_j=J_j}\delta_{x_1^{a_1}}\cdots\delta_{x_n^{a_n}}\otimes\delta_{x_1^{b_1}}\cdots\delta_{x_n^{b_n}} \\
&= \sum_{A+B=J}\delta_{x^A}\otimes\delta_{x^B}
\end{aligned}$$

as desired. The fact that Φ is a morphism of formal groups is now obvious. ∎

Example 3.8. We wish to take some time to describe the relevant features of the above theory in terms of the formal group corresponding to $\mathbb{T}_2$, the upper triangular 2×2 matrices of determinant one. Let us choose a coordinate system as follows:

$$\begin{pmatrix} 1+x & y \\ 0 & (1+x)^{-1} \end{pmatrix}$$

by this we mean that the co-multiplication laws, in these coordinates, are given by:

$$\begin{aligned}
x &\overset{m}{\longmapsto} x\otimes 1 + 1\otimes x + x\otimes x \\
y &\overset{m}{\longmapsto} (1+x)\otimes y + y\otimes(1+x)^{-1} \\
&\phantom{\overset{m}{\longmapsto}} = (1+x)\otimes y + y\otimes(1-x+x^2-x^3+\cdots).
\end{aligned}$$

Then x and y are geometric coordinates for the subgroups $\mathbb{G}_m$ and $\mathbb{G}_a$, respectively. Although we know that the underlying commutative group is given by $\mathbb{G}_m\times\mathbb{G}_a$, the PBW-theorem gives in fact a different set of coordinates for this group. Here, we need to consider the two choices of orderings $x<y$ and $x>' y$, and then a calculation shows that the resulting PBW-coordinates are:

$$\underset{x<y}{\begin{pmatrix} 1+x & y \\ 0 & (1+x) \end{pmatrix}} \quad \underset{x>' y.}{\begin{pmatrix} (1+x)^{-1} & y \\ 0 & (1+x)^{-1} \end{pmatrix}}$$

We give a complete description of $\mathbb{T}_2$. From the commutative theory, one knows that for $\mathbb{G}_m$ we have $\delta^p_{x^{p^r}}=\delta_{x^{p^r}}$ and for $\mathbb{G}_a$ we have $\delta^p_{y^{p^r}}=0$, thus we

only need concern ourselves with the commutator relations. Thus we compute:

$$\begin{aligned}
\delta_{x^{p^r}} \cdot \delta_{y^{p^s}} &= \sum_{a,b\geq 0} \langle m(x)^a m(y)^b, \delta_{x^{p^r}} \otimes \delta_{y^{p^s}} \rangle \delta_{x^a y^b} \\
&= \sum_{a,b\geq 0} \langle (x\otimes 1)^a (1\otimes y + x\otimes y)^b, \delta_{x^{p^r}} \otimes \delta_{y^{p^s}} \rangle \delta_{x^a y^b} \\
&= \sum_{a,b\geq 0} \langle \sum_{b'=0}^{b} \binom{b}{b'} x^{a+b'} \otimes y^b, \delta_{x^{p^r}} \otimes \delta_{y^{p^s}} \rangle \delta_{x^a y^b} \\
&= \begin{cases} \delta_{x^{p^r} y^{p^s}} & r < s \\ \delta_{x^{p^r} y^{p^s}} + \delta_{x^{p^r - p^s} y^{p^s}} & r \geq s. \end{cases}
\end{aligned}$$

A similar computation shows:

$$\delta_{y^{p^s}} \cdot \delta_{x^{p^r}} = \sum_{0\leq k\leq p^{r-s}} (-1)^k \delta_{x^{p^r - kp^s} y^{p^s}}$$

so that we have as commutator:

$$\pi_{\mathbf{c}}(\delta_{x^{p^r}}, \delta_{y^{p^s}}) = \begin{cases} 2\delta_{x^{p^r-p^s} y^{p^s}} - \sum_{2\leq k\leq p^{r-s}} (-1)^k \delta_{x^{p^r-kp^s} y^{p^s}} & r \geq s \\ 0 & r < s. \end{cases}$$

We note that the calculation is valid even for $p = 2$. The 2 in the commutator reflects the fact that the Lie algebra of $\mathbb{T}_2$ is abelian for $p = 2$.

Finally, we need to express $\pi_{\mathbf{c}}$ in multiplicative coordinates. Taking $x < y$, we only need to apply the following relation recursively:

$$\begin{aligned}
\delta_{x^{p^r}} \delta_{x^m y^{p^s}} &= (m_r + 1)\delta_{x^{m+p^r} y^{p^s}} + m_r \delta_{x^m y^{p^s}} \\
&\quad + \sum_{m\leq k\leq p^r - p^s} \binom{k}{k-m} \binom{m}{m + p^r - p^s - k} \delta_{x^k y^{p^s}}.
\end{aligned}$$

Here, $m = m_0 + \cdots m_r p^r$ is the p-adic expansion of m where we assume that $m_r < p - 1$. ∎

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